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Orthosummable Orthoalgebras

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We introduce notions of orthosummability and σ-orthosummability for orthoalgebras, which generalize the notions of orthocompleteness and σ-orthocompleteness for orthomodular posets, and we characterize such orthoalgebras in terms of their chains. We also show how to sum an infinite subset of an orthoalgebra, and we prove a generalized associative law for such sums.

1. INTRODUCTION

In 1936, Birkhoff and von Neumann (1936) considered the lattice of all closed subspaces of a separable infinite-dimensional Hilbert space as a mathematical model for a calculus of quantum logic by regarding such a lattice as a proposition system for a quantum mechanical entity. Such a lattice is usually called a *standard quantum logic*. Since then, there have been various attempts to abstract the standard quantum logics and their sets of *states* (σ-additive probability measures) and give a purely lattice-theoretic characterization of such logics. This has led to studying (σ-*complete*) *orthomodular lattices* and (σ-*orthocomplete*) *orthomodular posets* and their states as an abstraction of the standard quantum logics and their sets of states (Cook, 1978; D'Andrea and De Lucia, 1991; D'Andrea *et al*., 1991; Greechie, 1968; Gudder, 1965, 1988; Kalmbach, 1983, 1986; Lock, 1981; Mackey, 1963; Navara and Rogalewicz, 1991; Randall and Foulis, 1973, 1981).

Our abstraction of the standard quantum logic and its set of states is what we shall call a σ-*orthoalgebra* and its measures thereon. Orthoalgebras play an important role in the *empirical logic* approach to the mathematical foundation of quantum mechanics initiated by Foulis and Randall

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(Foulis and Randall, 1972; Foulis *et al.*, 1992; Randall and Foulis, 1973, 1981) because a *tensor product* can be defined for a large class of these algebras (Foulis and Bennett, 1993; Randall and Foulis, 1981), while no such product exists for orthomodular lattices or posets. Also, σ-orthoalgebras are generalizations of Boolean σ-algebras, standard quantum logics, σ-complete orthomodular lattices, and σ-orthocomplete orthomodular posets. Moreover, σ-orthoalgebras provide a mathematical basis for noncommutative measure theory in much the same way that σ-fields of sets provide a foundation for classical measure theory.

The main purpose of this paper is to study σ-orthoalgebras and their properties and thus set the stage to studying σ-additive states and observables on such orthoalgebras (which we shall do in a subsequent paper). In Section 2 we provide an in-depth review of the basic definitions and results from the newly developing theory of orthoalgebras which will be used in the following sections. In Section 3 we state and prove some fundamental lemmas about orthoalgebras, some of which will be used in the subsequent section. We also present a generalized associative law for orthogonal sums in orthoalgebras. In Section 4, we introduce the notion of *orthosummability* for orthoalgebras which can be considered as a natural extension of the notion of *orthocompleteness* for orthomodular posets in that it coincides with the latter notion if the underlying orthoalgebra happens to also be an orthomodular poset. Then we give a definition of a *σ-orthoalgebra* (see Definition 4.13) that is simpler and more natural than the rather strong definition (see the introduction to Section 4) given earlier by Younce (1987). Our definition does not depend on the blocks (maximal Boolean subalgebras) of the orthoalgebra, and it makes it easier to define σ-additive measures or states on such orthoalgebras. Furthermore, our definition of σ-orthoalgebra generalizes the notion of σ-orthocompleteness for orthomodular posets. We also give characterizations of orthosummable orthoalgebras, orthocomplete orthomodular posets, and σ-orthoalgebras in terms of their *chains* (see Theorems 4.4, 4.7, 4.9, and 4.14 and Corollaries 4.8 and 4.10).

About the time this paper was completed, Wilce and Feldman (1993) considered another definition (see the introduction to Section 4) of a σ-orthoalgebra, which turned out to be equivalent to our definition of σ-orthoalgebra (Theorem 4.14). However, it is not clear whether the uncountable version of their definition (i.e., Wilce and Feldman's definition of orthosummable orthoalgebra) is equivalent to our definition of orthosummable orthoalgebra.

For the most part the notation and symbols we use will be standard. If X is a set, the *cardinality* of X is denoted by $|X|$. The *power set* of X is denoted by $\mathscr{P}(X)$ and if $|X| = n$, then $\mathscr{P}(X)$ is sometimes denoted by $\mathbf{2}^n$. The symbols $\mathscr{F}(X)$, $c\mathscr{F}(X)$, and $\mathscr{I}(X)$ denote, respectively, the collection of all finite, cofinite, and infinite subsets of X. The symbols $\mathbf{R}$, $\mathbf{Z}$, and ω denote, respectively, the set of all real numbers, all integers, and all nonnegative integers.

2. DEFINITIONS AND PRELIMINARY RESULTS FROM THE THEORY OF ORTHOALGEBRAS

In this section, we shall review some definitions and results from the theory of orthoalgebras as well as from the theory of orthomodular lattices and posets. Most of the results of this section are known, and therefore we omit their proofs, which can be found in the references cited at the end of the paper. We also present some new results that will be used in the following sections.

A *partially ordered set* or simply a *poset* is a set P together with a binary relation $\leq$ on P which is reflexive, antisymmetric, and transitive. A poset $(P, \leq)$ for which P contains two distinguished elements 0 and 1, where 0 is the *smallest* element of P and 1 is the *largest* element of P, is called a *bounded* poset.

Let $(P, \leq)$ be a bounded poset, and let $x \in P$. An element $y \in P$ is called a *complement* of x in P if $x \vee y$ exists, $x \wedge y$ exists, $x \vee y = 1$, and $x \wedge y = 0$. If every element of P has a complement in P, then P is called a *complemented poset.* An *orthocomplementation* on P is a unary operation $': \mathrm{P} \to \mathrm{P}$ such that $\forall x, y \in P$:

1. $x \leq y \Rightarrow y' \leq x'$.
2. $x'' = x$ (where $x'' := (x')'$).
3. x' is a complement for x.

If $(P, \leq)$ is a complemented poset with an orthocomplementation $'$, then $(P, \leq, ')$ is called an *orthoposet* (or *orthocomplemented poset*). Unless confusion threatens, we will write P for $(P, \leq, ')$. It can be shown (Halmos, 1963) that the generalized De Morgan laws hold in any orthoposet.

An *orthoalgebra* (OA) is a quadruple $(L, \oplus, 0, 1)$, where L is a set containing two special elements 0, 1 and $\oplus$ is a partly defined binary operation on L that satisfies the following conditions $\forall p, q, r \in L$:

- (OA1) (*Commutativity*) If $p \oplus q$ is defined, *then* $q \oplus p$ is defined and $p \oplus q = q \oplus p$.
- (OA2) (*Associativity*) If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, *then* $p \oplus q$ is defined, $(p \oplus q) \oplus r$ is defined, and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

(OA3) (*Orthocomplementation*) For every $p \in L$ there exists a unique $q \in L$ such that $p \oplus q$ is defined and $p \oplus q = 1$.

(OA4) (*Consistency*) if $p \oplus p$ is defined, then $p = 0$.

Let $(L, \oplus, 0, 1)$ be an OA and let $p, q \in L$. We say that p is *orthogonal* to q in L and write $p \perp q$ if and only if $p \oplus q$ is defined in L. We define $p \leq q$ to mean that there exists $r \in L$ such that $p \perp r$ and $q = p \oplus r$. The unique element $q \in L$ corresponding to p in condition (OA3) above is called the *orthocomplement* of p and is written as p'. It can be easily proved (Foulis *et al.*, 1992) that L is partially ordered by $\leq$, that $0 \leq p \leq 1$ holds for all $p \in L$, that $p \perp q$ iff $p \leq q'$, and that $(L, \leq, ', 0, 1)$ is an orthoposet whenever L is an OA. Also, the following can be proved (Foulis *et al.*, 1992; Rüttimann, 1989; Gudder, 1988) $\forall p, q, r \in L$:

1. If $p \leq q$, then $q = p \oplus (p \oplus q')'$. This is called the *orthomodular identity* (OMI).
2. If $p \perp q$, then $p \oplus q$ is a *minimal* upper bound for $\{p, q\}$ in the poset L.
3. (*Cancellation law*) $p, q \perp r$ and $p \oplus r = q \oplus r \Rightarrow p = q$.
4. (*Cancellation law for inequalities*) $p, q \perp r$ and $p \oplus r \leq q \oplus r \Rightarrow p \leq q$.
5. $p \oplus 0 = p$.
6. $p \oplus q = 0 \Rightarrow p = q = 0$.

Let L be an OA. For p, q is called a *subelement* of q iff $p \leq q$. If p is a subelement of q, then, by the OMI, $q = p \oplus (p \oplus q')'$. In this case we define the *difference* of q and p in L by

$$q - p := (p \oplus q')'$$

The following elementary result is known. Nonetheless, we include a proof for which no reference seems to exist in the literature.

Lemma 2.1. Let L be an OA, and $a, b, x, y \in L$ be such that $a \leq x, b \leq y$, and $x \perp y$. Then:

1. $a \oplus b \leq x \oplus y$.
2. $a \oplus b = x \oplus y \Rightarrow a = x$ and $b = y$.

Proof. 1. We have $a \leq x \leq y' \leq b' \Rightarrow \{a, b, x - a, y - b\}$ is pairwise orthogonal. Also, $a \leq x \Rightarrow x = a \oplus (x - a)$ and $b \leq y \Rightarrow y = b \oplus (y - b)$. These and the computativity and associativity of $\oplus$ yield that $x \oplus y = a \oplus b \oplus (x - a) \oplus (y - b)$, which implies that $a \oplus b \leq x \oplus y$.

2. Assume that $a \oplus b = x \oplus y$. Then, as shown above, we have $x \oplus y = (a \oplus b) \oplus (x - a) \oplus (y - b) = x \oplus y \oplus (x - a) \oplus (y - b)$, which, using the cancellation law, yields that $(x - a) \oplus (y - b) = 0$. Thus, using

property 6 above, we have $x - a = 0$ and $y - b = 0$; hence $x = a$ and $y = b$. ■

An *orthomodular poset* (OMP) is an orthoalgebra P that satisfies the following condition: For $p, q \in P$, $p \perp q \Rightarrow p \vee q$ exists and $p \vee q = p \oplus q$. It can be shown (Foulis *et al.*, 1992; Gudder, 1988) that this condition is equivalent to the condition that for $p, q, r \in P$, $p \perp q \perp r \perp p \Rightarrow (p \oplus q) \perp r$. An *orthomodular lattice* (OML) is an OMP which is also a lattice. A *Boolean algebra* is a distributive OML.

Let L be an OA. A subset $A \subseteq L$ is called a *suborthoalgebra* (sub-OA) if $0, 1 \in A$ and, whenever $p, q \in A$ and $p \perp q$, it follows that $p' \in A$ and $p \oplus q \in A$.

Proposition 2.2. Let L be an OA and let $A \subseteq L$ be a sub-OA. For $p, q \in A$ put $p \oplus_A q := p \oplus q$ if $p \oplus q$ is defined. We have the following:

1. $(A, \oplus_A, 0, 1)$ is an OA.
2. $p \perp_A q$ iff $p \perp q$.
3. $p'^A = p'$.
4. $p \leq_A q$ iff $p \leq q$.

Proof. All parts are obvious except perhaps the "if" part of 4. So we prove this part. Suppose that $p \leq q$. Then $\exists r \in L$ such that $q = p \oplus r$. Then, by the OMI and the cancellation law, $r = (p \oplus q')' \in A$. Thus $p \leq_A q$. ■

Proposition 2.2 states that if L is an orthoalgebra and A is a sub-OA of L, then $\oplus_A$, $\perp_A$, $\leq_A$, and $'^A$ are the restrictions of $\oplus$, $-$, $\leq$, and $'$ to A, respectively. If $p, q \in A$, then the notation $p \vee^A q$ (resp., $p \wedge^A q$) stands for the supremum (resp., infimum) of the set $\{p, q\}$ as calculated in A.

Definition 2.3. Let L be an OA and $A \subseteq L$ be a sub-OA. Then A is called (1) a *sub-OMP* if $p, q \in A$, $p \perp q \Rightarrow p \vee^A q$ exists; (2) a *sub-OML* if $p, q \in A \Rightarrow p \vee^A q$ exists; (3) a *Boolean subalgebra* if it is a distributive sub-OML; and (4) a *block* if it is a maximal Boolean subalgebra under set-theoretic inclusion.

Let L be an OA and let $a, b \in L$. We say that a is *compatible with* b and write aCb iff $\{a, b\}$ is contained in a Boolean subalgebra of L. A subset $X \subset L$ is called *pairwise compatible* iff aCb $\forall a, b \in X$. A subset $X \subseteq L$ is called *jointly compatible* iff X is contained in a Boolean subalgebra. A subset $X \subseteq L$ is called *jointly orthogonal* iff it is pairwise orthogonal and jointly compatible. Let $M \subseteq L$. We define

$$J(L) := \{X \subseteq L\colon\ X \text{ is jointly orthogonal}\}$$

and

$$C(M) := \{x \in L\colon xCy\ \forall y \in M\}$$

Note that *the empty set and any singleton subject of L are jointly compatible* (*and jointly orthogonal* by default). Note also that if $p, q \in L$ and $p \perp q$, then one can easily check that $\{0, 1, p, q, p \oplus q, p', q', (p \oplus q)'\}$ is a Boolean subalgebra that contains $\{p, q\}$ and it has at most eight elements. Consequently, it follows that *every pairwise orthogonal subset of an orthoalgebra is pairwise compatible.*

Lemma 2.4. Let L be an orthoalgebra and let P be a sub-OMP of L. If $\{b_1, \ldots, b_n\} \subseteq P$ is pairwise orthogonal, then $b_1 \oplus \cdots \oplus b_n$ is defined (in L), $b_1 \vee^P \cdots \vee^P b_n$ exists (in P), and

$$b_1 \oplus \cdots \oplus b_n = b_1 \vee^P \cdots \vee^P b_n$$

Proof. We proceed by induction on n. Since P is an OMP, $b_1 \perp b_2 \Rightarrow b_1 \vee^P b_2 = b_1 \oplus b_2$. Assume $n > 1, b_1 \oplus \cdots \oplus b_{n-1}$ is defined, $b_1 \vee^P \cdots \vee^P b_{n-1}$ exists, and

$$b_1 \oplus \cdots \oplus b_{n-1} = b_1 \vee^P \cdots \vee^P b_{n-1}$$

Since $b_i \perp b_n \ \forall i \in \{1, \ldots, n-1\}$, we have $b_i \leq b_n' \ \forall \in \{1, \ldots, n-1\} \Rightarrow b_1 \vee^P \cdots \vee^P b_{n-1} \leq b_n' \Rightarrow (b_1 \oplus \cdots \oplus b_{n-1}) \perp b_n$. Hence $b_1 \oplus \cdots \oplus b_{n-1} \oplus b_n \in P$, $(b_1 \oplus \cdots \oplus b_{n-1}) \vee^P b_n$ exists and

$$b_1 \oplus \cdots \oplus b_n = (b_1 \oplus \cdots \oplus b_{n-1}) \vee^P b_n = b_1 \vee^P \cdots \vee^P b_{n-1} \vee^P b_n \quad \blacksquare$$

Now Lemma 2.4 justifies the following.

Convention 2.5. Let L be an OA and let $M = \{x_1, \ldots, x_n\} \in J(L)$. Then we shall write $\bigoplus M$ to mean $x_1 \oplus \cdots \oplus x_n$, which, by Lemma 2.4, equals $x_1 \vee^P \cdots \vee^P x_n$ for any sub-OMP P containing M.

The following lemma is known and its proof (Foulis *et al.*, 1992; Lock, 1981) is merely routine.

Lemma 2.6. Let L be an OA, $a, b \in L$. Then aCb iff there exists a triple $\{a_1, b_1, c\} \in J(L)$ such that

$$a = a_1 \oplus c \qquad \text{and} \qquad b = b_1 \oplus c$$

The following lemma, which will be used later, generalizes Lemma 15 of Kalmbach (1983, §4).

Lemma 2.7. Let B be a Boolean subalgebra of an OA L and $a, b \in B$. If $a \vee b$ exists, then $a \vee b \in B$. Moreover, if L is an OMP, then $a \vee b$ exists.

Proof. Let $a, b \in B$ and assume that $a \vee b$ exists. Since B is a Boolean subalgebra, Lemma 2.6 shows that there exists a triple $\{a_1, b_1, c\} \subseteq B$ such that $a_1 \perp b_1 \perp c \perp a_1, a = a_1 \oplus c$, and $b = b_1 \oplus c$. Moreover, $a = a_1 \oplus$

$c \perp b_1$; so $a \oplus b_1 = a_1 \oplus c \oplus b_1$. Also,

$$a \oplus b_1 = a \vee^B b_1 = a_1 \vee^B c \vee^B b_1$$
$$= a_1 \vee^B c \vee^B b_1 \vee^B c$$
$$= a \vee^B b$$

Thus we have $a, b_1 \leq a \vee b \leq a_1 \oplus c \oplus b_1 = a \oplus b_1 = a \vee^B b$; and, since $a \oplus b_1$ is a minimal upper bound for $\{a, b_1\}$ in L, we have $a \vee b = a \oplus b_1 = a \vee^B b \in B$.

Next, assume that L is an OMP and let $a, b \in B$. We need to show that $a \vee b$ exists. By Lemma 2.6, there exists a triple $\{a_1, b_1, c\} \subseteq B$ such that $a_1 \perp b_1 \perp c \perp a_1$ and $a = a_1 \vee c$ and $b = b_1 \vee c$. Moreover, the hypothesis that L is an OMP implies that $a = a_1 \oplus c \perp b_1$; so

$$a \oplus b_1 = a \vee b_= = a_1 \vee c \vee b_1$$
$$= (a_1 \vee c \vee b_1) \vee c$$
$$= (a_1 \vee c) \vee (b_1 \vee c)$$
$$= a \vee b \text{ exists} \quad \blacksquare$$

Note that Lemma 2.7 is no longer valid if the assumption that B is a Boolean subalgebra is weakened to assuming that B is a sub-OMP. For an example, let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $\mathscr{E}_8 := \{a \subseteq X: |a| \text{ is even}\}$. Replace L by $\mathscr{P}(X)$, B by $\mathscr{E}_8$ [which, as argued in Ramsay (1966), is a sub-OMP of $\mathscr{P}(X)$], a by $\{1, 3, 6, 8\}$, and b by $\{1, 2, 3, 4, 5, 6\}$ in Lemma 2.7 to see that it does not hold.

The following result is an immediate consequence of Lemma 2.7 and the De Morgan law.

Corollary 2.8. Let P be an OMP. For $a, b \in P$, $aCb \Rightarrow a \vee b$ and $a \wedge b$ both exist.

Lemma 2.9. The union of any chain of Boolean subalgebras of an orthoalgebra is a Boolean subalgebra.

Proof. Let L be an OA and let $\mathscr{B}$ be a chain of Boolean subalgebras of L under set-theoretic inclusion. We claim that $B =: \bigcup \mathscr{B}$ is a Boolean subalgebra. Clearly, B is closed under $'$ and contains 0 and 1. Let $a, b \in B$. We want to show that $a \vee^B b$ exists. Since $\mathscr{B}$ is a chain, there exists a $C \in \mathscr{B}$ such that $a, b \in C$. Note that $a \vee^C b$ is an upper bound for $\{a, b\}$. Let $u \in B$ be an upper bound of $\{a, b\}$. Since $\mathscr{B}$ is a chain, there exists a $D \in \mathscr{B}$ such that $C \cup \{u\} \subseteq D$ and C is a Boolean subalgebra of D. Now apply Lemma 2.7 to infer that $a \vee^C b = a \vee^D b \leq u$. It follows that $a \vee^B b$ exists and

$a \vee^{B} = a \vee^{C} b$. Hence B is a sub-OML of L. Now the distributivity of B follows from the facts that $\mathscr{B}$ is a chain and each of its members is distributive. ■

It follows from Lemma 2.9 and Zorn's Lemma that *every Boolean subalgebra of an orthoalgebra is contained in a block.*

Let L be an OA and $X \subseteq L$. Then the *sub-OA generated by* X, which we denote by $\Gamma(X)$, is the intersection of all sub-OA's of L that contain X.

Note that blocks of an OA always exist. In fact, every element p of an OA L is contained in at least one block, since the Boolean subalgebra generated by p (namely $\{0, 1, p, p'\}$), can be embedded into a maximal one. Thus every orthoalgebra can be regarded as a union of blocks. In this sense, an orthoalgebra is "locally Boolean."

We would like to point out that *our definition of a sub-OML of an OA* [as given in (2.3)] *is weaker than that of a sub-OML of an OML that is given in the literature* (Kalmbach, 1983). In fact, if L is an OML and $A \subseteq L$ is a sub-OML of L as an OA, then we do not require the joins (resp., meets) of elements of A as calculated in A to coincide with their joins (resp., meets) as calculated in L. For instance, consider the OML $L = G_{12}$ whose Hasse diagram is given in Figure 1. The subset $A = \{0, a, a', e, e', 1\}$ whose Hasse diagram is given in Figure 2 is a sub-OML of L as an OA. But it is not a sub-OML of L as an OML since $a \vee^{L} e = c' \neq 1 = a \vee^{A} e$. Also, this example shows that a sub-OML of an OA need not be a sub-OML of an OMP.

On the other hand, as shown in Lemma 2.7, our definition of a Boolean subalgebra of an OA coincides with the usual definition of a Boolean subalgebra of an OMP when the underlying OA happens to be an

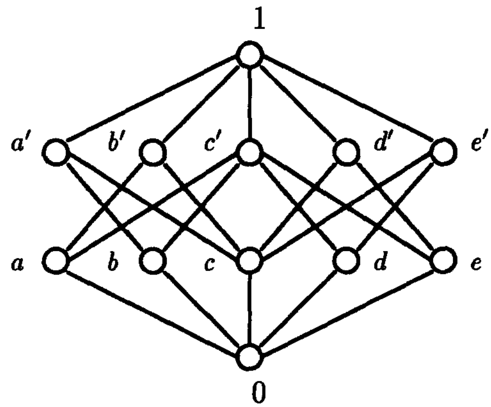

Fig. 1. G_{12}.

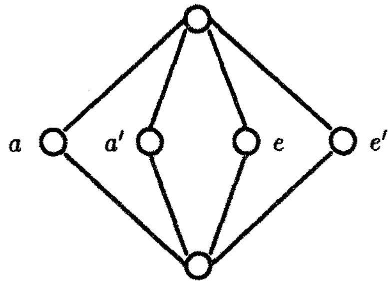

Fig. 2. $MO2$.

OMP. One reason we are considering this weak definition of a sub-OML is that over the years it has been a common practice in this field to define subobjects of different objects from different categories. For instance, sub-OMLs and Boolean subalgebras of an OMP are examples of such a practice. Another reason is that this weak definition serves our purposes.

Let L be an OA. The following can be shown (see, for example, Foulis *et al.*, 1992): *For* $p, q, r \in L$, $\{p, q, r\} \in J(L)$ *iff* $p \perp q$ *and* $p \oplus \perp r$, and L *is an OMP iff every three pairwise orthogonal elements of* L *are jointly orthogonal*. It can also be shown (Kalmbach, 1983; §4, Lemma 1) that *in an OML, pairwise compatible subsets are jointly compatible.*

Evidently, *every jointly orthogonal* (resp., *jointly compatible*) *subset of an OA is pairwise orthogonal* (resp., *pairwise compatible*) *but not conversely*, as can be easily seen from the Wright triangle example (Foulis *et al.*, 1992). *Even if L is an OMP, there may be subsets of L that are pairwise compatible, but not jointly compatible*, as Ramsay's example shows (Ramsay, 1966). Finally, *if L is a Boolean algebra, then by definition, any two elements of L are compatible*. Hence $C(L) = L$. However, *the converse need not be true*, as can be seen from the seven-point Fano projective plane example (Foulis *et al.*, 1992).

The following result gives an important necessary and sufficient condition for a sub-OML of an OA to be a Boolean subalgebra. It generalizes the well-known result that an OML is a Boolean algebra iff disjoint elements are orthogonal.

Theorem 2.10. Let L be an OA and let B be a sub-OML of L. Then B is a Boolean subalgebra of L iff the following condition holds:

$$\forall x, y \in B, \qquad x \wedge^B y = 0 \Rightarrow x \perp y$$

Proof. ($\Rightarrow$): Assume that B is a Boolean subalgebra, and let $x, y \in B$ be such that $x \vee^B y = 0$. Then, by the distributive law,

$$\begin{aligned} x &= (x \wedge^B y) \vee^B (x \wedge^B y') \\ &= x \wedge^B y' \end{aligned}$$

which implies that $x \leq_B y'$ and hence $x \leq y'$, i.e., $x \perp y$.

($\Leftarrow$): Assume that the sub-OML B satisfies the stated condition. We need to prove that B is distributive. Let $x, y, z \in B$. Clearly,

$$x \wedge^B (y \vee^B z) \geq (x \wedge^B y) \vee^B (x \wedge^B z) \tag{2.1}$$

To show equality holds in (2.1), it suffices to show, thanks to the OMI, that

$$x \wedge^B (y \vee^B z) \wedge^B (x' \vee^B y') \wedge^B (x' \vee^B z') = 0 \tag{2.2}$$

To this end, let $b \in B$ be such that

$$b \leq \{x, y \vee^B z, x' \vee^B y', x' \vee^B z'\}$$

Since $y \wedge^B [x \wedge^B (x' \vee^B y')] = 0$, the hypothesized condition implies that $x \wedge^B (x' \vee^B y') \perp y$. It follows that $b \leq x \wedge^B (x' \vee^B y') \leq y'$. Similarly, $b \leq z'$. Thus $b \leq y' \wedge^B z' = (y \vee^B z)'$. This and the assumption that $b \leq y \vee^B z$ show that $b = 0$. Now (2.2) follows. ■

Using Varadarajan's Lemma (Varadarajan, 1962, Proposition 3.8), it is not difficult to show that if P is an OMP and $x \in P$, then $C(x)$ is a sub-OMP of P. We conclude this section with the following result, which will be used in the following sections.

Corollary 2.11. Let P be an OMP and $X \in \mathscr{F}(P)$ be pairwise orthogonal. Then $\Gamma(X)$ is a Boolean subalgebra of P and hence X is jointly orthogonal.

Proof. Write $X = \{x_1, \ldots, x_n\}$. We may assume that $0 \notin X$. We proceed by induction on n. If $n = 1$, then $X = \{x_1\}$ and $\Gamma(\{x_1\}) = \{0, 1, x, x'\}$ is a Boolean subalgebra that has at most four elements.

Assume that $n > 1$ and $\Gamma(\{x_1, \ldots, x_{n-1}\})$ is a Boolean subalgebra. Since $x_i \perp x_n$ $\forall i \in \{1, \ldots, n-1\}$, we have $\{x_1, \ldots, x_{n-1}\} \subseteq C(x_n)$. This and the fact that $C(x_n)$ is a sub-OMP of P imply that $\Gamma(\{x_1, \ldots, x_{n-1}\}) \subseteq C(x_n)$. Now, by Lemma 9 of Ramsay (1966),

$$\Gamma(\Gamma(\{x_1, \ldots, x_{n-1}\}) \cup \{x_n\}) = \Gamma(\{x_1, \ldots, x_n\})$$

is a Boolean subalgebra. This completes the induction. ■

3. KEY LEMMAS

In this section we state and prove some key results about orthoalgebras. Some of these results will be used in proving theorems in the following section, and some have their own interest and may prove to be useful in the study of orthoalgebras. We start with the following lemma, which generalizes Lemma 2.14 of Rüttimann (1989).

Lemma 3.1. Let L be an OA, let $X \subseteq L$ be such that $\mathscr{F}(X) \subseteq J(L)$, and let $M, N \in \mathscr{F}(X)$. Then we have the following:

(i) $M \cap N \subseteq \{0\}$ iff $\bigoplus M \perp \bigoplus N$. In either case,

$$\bigoplus M \oplus \bigoplus N = \bigoplus (M \cup N)$$

(ii) $M \subset N$ iff $\bigoplus M \perp (\bigoplus N)'$. In either case,

$$\bigoplus M \oplus (\bigoplus N)' = [\bigoplus (N \backslash M)]'$$

(iii) $(\bigoplus M)' \perp (\bigoplus N)'$ iff $(\bigoplus N)' = \bigoplus (M \backslash N)$. In either case

$$(\bigoplus M) \oplus (\bigoplus N)' = [\bigoplus (M \cap N)]'$$

Proof. It is essentially the same as the proof of Lemma 2.14 of Rüttimann (1989). ■

Lemma 3.2. Let L be an OA and let $X \subseteq L$ be such that $\mathscr{F}(X) \subseteq J(L)$. Then

$$B(X) := \{\bigoplus M : M \in \mathscr{F}(X)\} \cup \{(\bigoplus N)' : N \in \mathscr{F}(X)\}$$

(where $\bigoplus \varnothing := 0$, and $\bigoplus \{x\} = x \ \forall x \in X$) is a sub-OMP (see Corollary 2.8) containing X.

Proof. By its definition, $B(X)$ is closed under $'$; and, by Lemma 3.1, $B(X)$ is closed under the $\oplus$ of orthogonal pairs. Also, $0 = \bigoplus \varnothing \in B(X)$. Now to prove that $B(X)$ is a sub-OMP, we need to show that

$$a, b \in B(X), \qquad a \perp b \ \Rightarrow \ a \vee^{B(X)} b \quad \text{exists}$$

This follows immediately from Lemma 3.1, which yields that $a \vee^{B(X)} b$ exists and $a \vee^{B(X)} b = a \oplus b$. ■

Lemma 3.3. Let L, X, and $B(X)$ be as in the statement of Lemma 3.2. Then aCb for all $a, b \in B(X)$.

Proof. It is enough to show that $(\bigoplus M)C(\bigoplus N)\ \forall M, N \in \mathscr{F}(X)$. For

$$(\bigoplus M)C(\bigoplus N) \ \Rightarrow \ (\bigoplus M)C(\bigoplus N)'$$
$$\Rightarrow \ (\bigoplus M)'C(\bigoplus N)' \ \Rightarrow \ aCb \qquad \forall a, b \in B(X)$$

Now since

$$M = (M \backslash N) \cup (M \cap N) \qquad \text{and} \qquad N = (N \backslash M) \cup (M \cap N)$$

we have

$$\bigoplus M = \bigoplus (M\backslash N) \oplus \bigoplus (M \cap N)$$
$$\bigoplus N = \bigoplus (N\backslash M) \oplus \bigoplus (M \cap N)$$

and

$$\bigoplus (M\backslash N) \perp \bigoplus (N\backslash M) \perp \bigoplus (N\backslash M) \perp \bigoplus (M\backslash N)$$

Since $\{\bigoplus (M\backslash N), \bigoplus (N\backslash M), \bigoplus (M \cap N)\} \subseteq B(X)$ and, by Lemma 3.2, $B(X)$ is a sub-OMP, Corollary 2.11 shows that $\{\bigoplus (M\backslash N), \bigoplus (N\backslash M), \bigoplus (M \cap N)\}$ is jointly orthogonal. Now, by Lemma 2.6, $(\bigoplus M)C(\bigoplus N)$. ■

The following theorem gives a sufficient (and, trivially, a necessary) condition for a subset of an OA to be jointly orthogonal.

Theorem 3.4. Let L be an OA and let $X \subseteq L$. If every finite subset of X is jointly orthogonal, then X is jointly orthogonal.

Proof. Assume that $\mathscr{F}(X) \subseteq J(L)$. Form $B(X)$ as in the statement of Lemma 3.2. By Lemma 3.2, $B(X)$ is a sub-OMP, and, by Lemma 3.3, it is pairwise compatible. Now, using the fact (Lock, 1981, Theorem 4.5.6) that an OA is a Boolean algebra iff it is an OMP and pairwise compatible, we infer that $B(X)$ is a Boolean subalgebra containing X. ■

Corollary 3.5. Let P be an OMP and let $X \subseteq P$. Then X is jointly orthogonal iff X is pairwise orthogonal.

Proof. By Corollary 2.11, every finite pairwise orthogonal subset of X is jointly orthogonal. Now apply Theorem 3.4. ■

The following theorem is analogous to Theorem 3.4. It gives a sufficient condition for a subset of an OA to be jointly compatible.

Theorem 3.6. Let L be an OA and let $X \subseteq L$. If every finite subset of X is jointly compatible, then X is jointly compatible.

Proof. Assume that every finite subset of X is jointly compatible. Let $F \in \mathscr{F}(X)$. Then, by assumption, F is contained in a Boolean subalgebra of L. Hence F is contained in a smallest Boolean subalgebra, namely $\Gamma(F)$, the Boolean subalgebra generated by F. It is well known (Halmos, 1963) that $\Gamma(F)$ is isomorphic to the Boolean algebra $\mathbf{2}^k$ (of 2^k elements), where $k = 2^{|F|}$. Hence it follows that

$$F, G \in \mathscr{F}(X), \qquad F \subseteq G \;\Rightarrow\; \Gamma(F) \quad \text{is isomorphic to a subalgebra of } \Gamma(G)$$

Now, as argued in the proof of Lemma 2.9, the last implication shows that

$\bigcup_{F \in \mathscr{F}(X)} \Gamma(F)$ is a Boolean subalgebra containing X. Therefore X is jointly compatible. ■

Definition 3.7. Let L be an OA and let $X \in J(L)$. If the supremum of the set $\{\bigoplus F: F \in \mathscr{F}(X)\}$ exists in L, then we define

$$\bigoplus X := \bigvee_{F \in \mathscr{F}(X)} \bigoplus F$$

Note that if L is an OA, $a, b \in L$, and $a \perp b$, then $\{a, b\} \in J(L)$ and so

$$\bigoplus \{a, b\} = \bigvee \{0, a, b, a \oplus b\} = a \oplus b$$

Similarly, for any $\{a_1, \ldots, a_n\} \in J(L)$

$$\bigoplus \{a_1, \ldots, a_n\} = a_1 \oplus \cdots \oplus a_n$$

Thus the above definition of $\bigoplus$ is consistent with the partial binary operation $\oplus$ that is defined on L and serves as a natural extension of its (finitary) orthogonal sum that we established in Section 2 (see Convention 2.5).

Lemma 3.8. Let L be an OA and let $X, Y \subseteq L$.

(i) If $X \cap Y \subseteq \{0\}$, $X \cup Y \in J(L)$, and $\bigoplus X$, $\bigoplus Y$ both exist, then $\bigoplus X \perp \bigoplus Y$.

(ii) If, in addition to the hypotheses of (i), $\bigoplus (X \cup Y)$ exists, then

$$(\bigoplus X) \oplus (\bigoplus Y) = \bigoplus (X \cup Y)$$

Proof. (i) Clearly $\bigoplus F \perp \bigoplus G$ $\forall F \in \mathscr{F}(X)$ and $\forall G \in \mathscr{F}(Y)$. So the sets $\{\bigoplus F: F \in \mathscr{F}(X)\}$ and $\{\bigoplus G: G \in \mathscr{F}(Y)\}$ are pairwise orthogonal. Fix $F \in \mathscr{F}(X)$. Then

$$\bigoplus F \leq (\bigoplus G)' \qquad \forall G \in \mathscr{F}(Y)$$

so, as $\bigvee_{G \in \mathscr{F}(Y)} \bigoplus G$ exists, the generalized De Morgan law implies that

$$\bigoplus F \leq \bigwedge_{G \in \mathscr{F}(Y)} (\bigoplus G)' = \left(\bigvee_{G \in \mathscr{F}(Y)} \bigoplus G\right)'$$

Since this holds $\forall F \in \mathscr{F}(X)$, we obtain

$$\bigvee_{F \in \mathscr{F}(X)} \bigoplus F \leq \left(\bigvee_{G \in \mathscr{F}(Y)} \bigoplus G\right)'$$

and therefore $\bigoplus X \perp \bigoplus Y$.

(ii) Clearly,

$$\bigoplus F_1 \leq \bigvee_{F \in \mathscr{F}(X)} \bigoplus F \qquad \forall F_1 \in \mathscr{F}(X)$$

$$\bigoplus F_2 \leq \bigvee_{G \in \mathscr{F}(Y)} \bigoplus G \qquad \forall F_2 \in \mathscr{F}(Y)$$

So, as $\bigvee_{F\in\mathscr{F}(X)} \bigoplus F \perp \bigvee_{G\in\mathscr{F}(Y)} \bigoplus G$, part 1 of Lemma 2.1 implies that $\forall F_1 \in \mathscr{F}(X)$, $\forall F_2 \in \mathscr{F}(Y)$, and $\forall H \in \mathscr{F}(X \cup Y)$

$$(\bigoplus F_1) \oplus (\bigoplus F_2) \leq \left(\bigvee_{F\in\mathscr{F}(X)} \bigoplus F\right) \oplus \left(\bigvee_{G\in\mathscr{F}(Y)} \bigoplus G\right)$$

$$\Rightarrow \bigoplus H \leq \left(\bigvee_{F\in\mathscr{F}(X)} \bigoplus F\right) \oplus \left(\bigvee_{G\in\mathscr{F}(Y)} \bigoplus G\right)$$

$$\Rightarrow \bigvee_{H\in\mathscr{F}(X\cup Y)} \bigoplus H \leq \left(\bigvee_{F\in\mathscr{F}(X)} \bigoplus F\right) \oplus \left(\bigvee_{G\in\mathscr{F}(X)} \bigoplus G\right)$$

On the other hand, it is clear that

$$\bigvee_{F\in\mathscr{F}(X)} \bigoplus F, \quad \bigvee_{G\in\mathscr{F}(Y)} \bigoplus G \leq \bigvee_{H\in\mathscr{F}(X\cup Y)} \bigoplus H$$

so, as $(\bigvee_{F\in\mathscr{F}(X)} \bigoplus F) \oplus (\bigvee_{G\in\mathscr{F}(Y)} \bigoplus G)$ is a minimal upper bound for the set $\{\bigvee_{F\in\mathscr{F}(X)} \bigoplus F, \bigvee_{G\in\mathscr{F}(Y)} \bigoplus G\}$, we have

$$\left(\bigvee_{F\in\mathscr{F}(X)} \bigoplus F\right) \oplus \left(\bigvee_{G\in\mathscr{T}(Y)} \bigoplus G\right) = \bigvee_{H\in\mathscr{F}(X\cup Y)} \bigoplus H \quad \blacksquare$$

The following corollary is simply a paraphrasing of part (ii) of Lemma 3.8.

Corollary 3.9. Let L be an OA and let $(x_i)_i, (y_j)_j \subseteq L$. If $\bigvee_i x_i$, $\bigvee_j y_j$, and $\bigvee_{i,j}(x_i \oplus y_j)$ all exist in L, and if $\bigvee_i x_i \perp \bigvee_j y_j$, then

$$\left(\bigvee_i x_i\right) \oplus \left(\bigvee_j y_j\right) = \bigvee_{i,j} (x_i \oplus y_j)$$

Lemma 3.10. Let L be an OA and let $Y \in J(L)$. If $T_1, T_2 \subseteq Y$ are such that $\bigoplus T_1$, $\bigoplus T_2$ both exist and $\bigoplus T_1 = \bigoplus T_2$, then $T_1 = T_2$.

Proof. We may assume that $0 \notin Y$. Suppose that $\exists t \in T_2 \backslash T_1$. Since $T_1 \cup \{t\} \in J(L)$, there exists a block $B \supseteq T_1 \cup \{t\}$. Then $\forall F \in \mathscr{F}(T_1)$, $\bigoplus F = \bigvee^B F \leq t'$, which implies that $\bigoplus T_1 = \bigvee_{F\in\mathscr{F}(T_1)} \bigoplus F \leq t'$ and therefore $t \perp \bigoplus T_1 = \bigoplus T_2$. As $t \leq \bigoplus T_2$ (since $t \in T_2$), this implies that $t \perp t$ and hence $t = 0$, a contradiction. $\blacksquare$

The following theorem will be used later when we study orthosummable orthoalgebras and σ-orthoalgebras. It generalizes Lemma 2 of Ramsay (1966).

Theorem 3.11. Let L be an OA. If $X \in J(L)$ is such that $\bigoplus T$ exists in L for all $T \subseteq X$ and $\bigoplus X = 1$, then

$$B(X) := \{\bigoplus T: T \subseteq X\}$$

is a complete Boolean subalgebra of L containing X and the mapping $T \mapsto \bigoplus T: \mathscr{P}(X) \to L$ is an isomorphism of $\mathscr{P}(X)$ onto $B(X)$.

Proof. Since for every $T \subseteq X, (X \backslash T) \cap T \subseteq \{0\}$, Lemma 3.8 implies that

(i) $(\bigoplus T)' = \bigoplus (X \backslash T) \in B(X) \qquad \forall T \subseteq X$

Also, Lemma 3.8 implies that

(ii) $\bigoplus T_1 \perp \bigoplus T_2 \Rightarrow (\bigoplus T_1) \oplus (\bigoplus T_2) = \bigoplus (T_1 \cup T_2) \in B(X)$

and, by the hypothesis, we have

(iii) $\bigoplus X = 1 \in B(X)$

It follows from (i)–(iii) that $B(X)$ is a sub-OA of L.

Claim. $T \mapsto \bigoplus T: \mathscr{P}(X) \to B$ is an isomorphism.

Note first that, by Lemma 3.10, the mapping $\bigoplus$ is injective, and it is clear that it is surjective. Note next that, by (ii), the mapping $\oplus$ is a morphism of OAs. Now to prove the claim, it suffices to show, using Theorem 2.9 of Habil (1993), that $\forall T, S \subseteq X$,

$$\bigoplus T \leq \bigoplus S \Rightarrow T \subseteq S$$

To this end, suppose that $\bigoplus T \leq \bigoplus S$. We may assume that $T, S \subseteq X \backslash \{0\}$. Then, since $B(X)$ is a sub-OA, there exists $R \subseteq X$ such that $\bigoplus R \perp \bigoplus T$ and

$$\bigoplus S = \bigoplus T \oplus \bigoplus R \overset{\text{(ii)}}{=} \bigoplus (T \cup R)$$

so Lemma 3.10 implies that $S = T \cup R$ and hence $T \subseteq S$. This proves the claim.

Now since $\mathscr{P}(X)$ is a Boolean algebra, the above claim shows that $B(X)$ is a Boolean algebra. ■

The remaining part of this section, which is quite involved, is devoted to proving a *generalized associative law* for the operation $\oplus$ in orthoalgebras (see Theorem 3.16 below). As we shall see, this result generalizes a similar result (Foulis and Bennett, 1993, Lemma 2.9; Wilce and Feldman, 1993, Lemma 1.4) and will prove to be useful when we study orthosummable orthoalgebras in the next section. We first make a digression to study "interval orthoalgebras." Let L be an OA. Recall the notation that for $x \in L$,

$$[0, x] = \{a \in L: a \leq x\}$$

Note that the interval $[0, x]$, with the orthogonality relation being the restriction of the orthogonality relation on L, need not be an OA under the $\oplus$ of L. For example, let L be the Wright triangle, where the atoms are labeled as in Figure 3. Then $\{a, b, c, d\} \subseteq [0, x]$ and $b \perp c$, but $b \oplus c = e' \nleq x$.

However, we can make each interval $[0, x]$ in L into an OA as follows.

Definition 3.12. Let L be an OA and let $x \in L$. We define an orthogonality relation $\perp_x$ on $[0, x]$ as follows:

$$a \perp_x b \qquad \text{iff } a \perp b \text{ in } L \text{ and } a \oplus b \leq x \qquad \forall a. b \in [0, x]$$

This induces a $\oplus$ on $[0, x]$, which we denote by $\oplus_x$, where for $a, b \in [0, x]$, $a \oplus_x b$ is defined in $[0, x]$ iff $a \perp_x b$. It can be easily checked that with this *induced addition*, $([0, x], \oplus_x, 0, x)$ is an OA. We call such an OA an *induced orthoalgebra*. Note also that the orthocomplementation $p \mapsto p': L \to L$ induces a *relative orthocomplementation* $a \mapsto a'^x: [0, x] \to [0, x]$ which is given by

$$a'^x := x - a \qquad \forall a \in [0, x]$$

Furthermore, $\leq_{[0,x]}$ is the restriction of $\leq$ to $[0, x]$. Indeed, suppose that $a, b \in [0, x]$ and $a \leq b$ in L. Then $\exists c \in L$ such that $c \perp a$ and $a \oplus c = b$. Now $c \leq a \oplus c = b \leq x \Rightarrow c \in [0, x]$. This shows that $a \leq_{[0,x]} b$. On the other hand, if $a, b \in [0, x]$ and $a \leq_{[0,x]} b$, then $a \leq b$ in L is obvious.

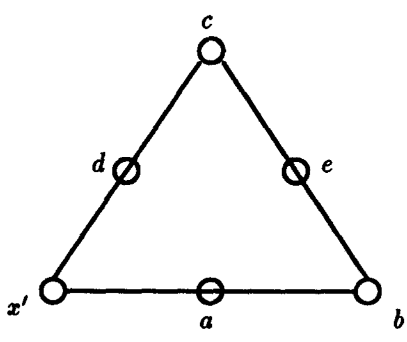

Fig. 3. Greechie diagram of the Wright triangle.

From now on, each interval $[0, x]$ $(x \in L)$ will be considered to be an induced orthoalgebra.

Lemma 3.13. Let L be an OA and let $x \in L$. If B is a Boolean subalgebra of $[0, x]$, then we have

$$a, b \in B \quad \text{and} \quad a \perp b \Rightarrow a \oplus b \leq x \qquad (\text{i.e., } a \perp_x b)$$

Proof. Let B be a Boolean subalgebra of $[0, x]$ and let $a, b \in B$. Clearly, $a \perp b \Rightarrow a \wedge^L b = 0 \Rightarrow a \wedge^{[0,x]} b = 0$ (because $\leq_{[0,x]} = \leq|_{[0,x]}) \Rightarrow a \wedge^B b = 0$ (because $\leq_B = \leq_{[0,x]}|_B) \Rightarrow a \perp_B b$ (by Theorem 2.10, since B is a Boolean subalgebra of $[0, x]) \Rightarrow a \perp_x b$ (by Proposition 2.2) $\Leftrightarrow a \oplus b \leq x$. ■

Let L be an OA and let $x, y \in L$ be such that $x \perp y$. If B is a sub-OA of $[0, x]$ and D is a sub-OA of $[0, y]$, then $b \perp d$ $\forall b \in B$ and $\forall d \in D$. We define

$$B \oplus D := \{b \oplus d: b \in B, d \in D\}$$

The following theorem is a key to proving the main result of this section.

Theorem 3.14. Let L be an OA and let $x, y \in L$ be such that $x \perp y$. If B is a Boolean subalgebra of $[0, x]$ and D is a Boolean subalgebra of $[0, y]$, then $B \oplus D$ is a Boolean subalgebra of $[0, x \oplus y]$.

Proof. Note that if $b \oplus d \in B \oplus D$, then $b \leq x$ and $d \leq y$. So, as $x \perp y$, part 1 of Lemma 2.1 implies that $b \oplus d \leq x \oplus y$. Thus $B \oplus D \subseteq [0, x \oplus y]$.

First, we claim that $B \oplus D$ is a sub-OA of $[0, x \oplus y]$. To see this, let $b_1, b_2 \in B$ and $d_1, d_2 \in D$ be such that $b_1 \oplus d_1, b_2 \oplus d_2 \in B \oplus D$, and $b_1 \oplus d_1 \perp_{(x \oplus y)} b_2 \oplus d_2$. This means that $b_1 \oplus d_1, b_2 \oplus d_2, (b_1 \oplus d_1) \oplus (b_2 \oplus d_2) \leq x \oplus y$. By the commutativity and associativity of $\oplus$, we have $(b_1 \oplus d_1) \oplus (b_2 \oplus d_2) = (b_1 \oplus b_2) \oplus (d_1 \oplus d_2)$. This implies that $b_1 \perp b_2$ and $d_1 \perp d_2$. Hence, by Lemma 3.13, $b_1 \perp_x b_2$ and $d_1 \perp_y d_2$. Since B (resp., D) is a Boolean subalgebra of $[0, x]$ (resp., $[0, x]$), it follows that $b_1 \oplus b_2 \in B$ and $d_1 \oplus d_2 \in D$. Therefore

$$(b_1 \oplus d_1) \oplus (b_2 \oplus d_2) \in B \oplus D$$

For any $b \oplus d \in B \oplus D$, we have

$$x \oplus y = (b \oplus (x - b)) \oplus (d \oplus (y - d))$$

$$= (b \oplus d) \oplus (x - b) \oplus (y - d)$$

by the commutativity and associativity of $\oplus$, which implies that

$$(b \oplus d)'^{(x \oplus y)} = (x - b) \oplus (y - d) \in B \oplus D$$

Furthermore, $0 = 0 \oplus 0 \in B \oplus D$. Thus $B \oplus D$ is a sub-OA of $[0, x \oplus y]$.

Second, we claim that $B \oplus D$ is isomorphic to the Boolean algebra $B \times D$, where $B \times D = \{(b, d): b \in B, d \in D\}$. To see this, define $\phi: B \times D \to B \oplus D$ by

$$\phi(b, d) = b \oplus d \qquad \forall (b, d) \in B \times D$$

We have the following:

(i) ϕ is surjective. This is obvious.

(ii) ϕ is injective. To see this, suppose that $b_1 \oplus d_1 = b_2 \oplus d_2$ for some $b_1, b_2 \in B$ and $d_1, d_2 \in D$. We must show that $b_1 = b_2$ and $d_1 = d_2$. Using the commutativity and associativity of $\oplus$, we have $x = b_1 \oplus (x - b_1)$ and $y = d_1 \oplus (y - d_1)$; so

$$\begin{aligned} x \oplus y &= b_1 \oplus d_1 \oplus (x - b_1) \oplus (y - d_1) \\ &= b_2 \oplus d_2 \oplus (x - b_1) \oplus (y - d_1) \\ &= (b_2 \oplus (x - b_1)) \oplus (d_2 \oplus (y - d_1)) \end{aligned}$$

This implies that $b_2 \perp x - b_1$ and $d_2 \perp y - d_1$. Since $b_2, x - b_1 \in B$, and B is a Boolean subalgebra of $[0, x]$, Lemma 3.13 implies that $b_2 \oplus (x - b_1) \leq x$. Similarly, $d_2 \oplus (y - d_1) \leq y$. Now part 2 of Lemma 2.1 implies that $b_2 \oplus (x - b_1) = x$ and $d_2 \oplus (y - d_1) = y$. Hence

$$b_2 \oplus (x - b_1) = b_1 \oplus (x - h) \qquad \text{and} \qquad d_2 \oplus (y - d_1) = d_1 \oplus (y - d_1)$$

So the cancellation law implies that $b_2 = b_1$ and $d_2 = d_1$. Therefore ϕ is one-to-one.

(iii) ϕ is a morphism. To see this, suppose that $(b_1, d_2) \perp (b_2, d_2)$ in $B \times D$. This means that $b_1 \perp b_2$ in B and $d_1 \perp d_2$ in D. Hence $b_1 \perp_x b_2, d_1 \perp_y d_2$, and so $b_1 \leq b_2'^x = x - b_2$ and $d_1 \leq d_2'^y = y - d_2$. Since $x \perp y \Rightarrow x - b_2 \perp y - d_2$, part 1 of Lemma 2.1 now shows that $b_1 \oplus d_1 \leq (x - b_2) \oplus (y - d_2) = (b_2 \oplus d_2)'^{(x \oplus y)}$, i.e., $\phi(b_1, d_1) \perp_{(x \oplus y)} \phi(b_2, d_2)$. Moreover, the commutativity and associativity of $\oplus$ yield that

$$\begin{aligned} \phi((b_1, d_1) \oplus (b_2, d_2)) &= \phi(b_1 \oplus b_2, d_1 \oplus d_2) \\ &= (b_1 \oplus d_1) \oplus (b_2 \oplus d_2) \\ &= \phi(b_1, d_1) \oplus \phi(b_2, d_2) \end{aligned}$$

This and the fact that $\phi(x, y) = x \oplus y$ show that ϕ is a morphism of OAs.

(iv) ϕ^{-1} preserves $\leq$. To see this, suppose that $b_1 \oplus d_1, b_2 \oplus d_2 \in B \oplus D$, and $b_1 \oplus d_1 \leq b_2 \oplus d_2$. Then, since $B \oplus D$ is a sub-OA, there exists $b \oplus d \in B \oplus D$ such that $(b \oplus d) \perp (b_1 \oplus d_1)$ and $b_2 \oplus d_2 = (b_1 \oplus d_1) \oplus (b \oplus d) = (b_1 \oplus b) \oplus (d_1 \oplus d) \Rightarrow \phi(b_2, d_2) = \phi(b_1 \oplus b, d_1 \oplus d)$. So, as ϕ is one-to-one, this gives $(b_2, d_2) = (b_1 \oplus b, d_1 \oplus d) \Rightarrow b_2 = b_1 \oplus b$ and $d_2 = d_1 \oplus d \Rightarrow b_1 \leq b_2$ and $d_1 \leq d_2 \Rightarrow (b_1, d_1) \leq (b_2, d_2)$.

Now combining (i)–(iv) together, it follows from Theorem 2.9 of Habil (1993) that ϕ is an isomorphism. Thus, as $B \times D$ is a Boolean algebra, we conclude that $B \oplus D$ is a Boolean subalgebra of $[0, x \oplus y]$. ∎

Here is an important application of Theorems 3.11 and 3.14.

Lemma 3.15. Let L be an OA and let $X, Y \in J(L)$ be such that $\bigoplus X$ and $\bigoplus Y$ exist, $X \cup \{(\bigoplus X)'\}$, $Y \cup \{(\bigoplus Y)'\} \in J(L)$, $\bigoplus \Delta$ exists for all $\Delta \subseteq X \cup \{(\bigoplus X)'\}$, and $\bigoplus \Gamma$ exists for all $\Gamma \subseteq Y \cup \{(\bigoplus Y)'\}$. If $\bigoplus X \perp \bigoplus Y$, then $X \cap Y \subseteq \{0\}$ and $X \cup Y \in J(L)$.

Proof. First, if $z \in X \cap Y$, then the hypothesis that $\bigoplus X \perp \bigoplus Y$ and the fact that subelements of orthogonal elements in any OA are also orthogonal imply that $z \perp z \Rightarrow z = 0$. Therefore $X \cap Y \subseteq \{0\}$.

Second, we show that $X \cup Y \in J(L)$. Let $\bar{X} := X \cup \{(\bigoplus X)'\}$ and $\bar{Y} := Y \cup \{(\bigoplus Y)'\}$. By the hypothesis, $\bar{X}, \bar{Y} \in J(L)$ and $\bigoplus \bar{X}$, $\bigoplus \bar{Y}$ exist. Hence, by Lemma 3.8, $\bigoplus \bar{X} = (\bigoplus X) \oplus (\bigoplus X)' = 1$ and $\bigoplus \bar{Y} = (\bigoplus Y) \oplus (\bigoplus Y)' = 1$. Now two applications of Theorem 3.11 yield that there exist Boolean subalgebras $B(\bar{X})$, $B(\bar{Y})$ of L such that $X, \bigoplus X \in B(\bar{X})$ and $Y, \bigoplus Y \in B(\bar{Y})$. Let

$$B_1 := B(\bar{X}) \cap [0, \bigoplus X] \qquad \text{and} \qquad B_2 := B(\bar{Y}) \cap [0, \bigoplus Y]$$

We claim that B_1 is a Boolean subalgebra of $[0, \bigoplus X]$, B_2 is a Boolean algebra of $[0, \bigoplus Y]$, $X \subseteq B_1$, and $Y \subseteq B_2$. To see this, note first that $0, \bigoplus X \in B_1$. If $a, b \in B_1$ and $a \perp_{\oplus X} b$, then $a \perp b$ and $a \oplus b \le \bigoplus X$. Also, $a, b \in B(\bar{X}) \Rightarrow a \oplus b \in B(\bar{X})$. Therefore $a \oplus_{\oplus X} b = a \oplus b \in B_1$. If $a \in B_1$, then $a \in B(\bar{X})$ and $a \le \bigoplus X$. So, as $a, \bigoplus X \in B(\bar{X})$, $\bigoplus X - a \in B(\bar{X})$. Hence, as $\bigoplus X - a \le \bigoplus X$, we obtain

$$a'^{\oplus X} = \bigoplus X - a \in B(\bar{X}) \cap [0, \bigoplus X] = B_1$$

Thus we have shown that B_1 is a sub-OA of $[0, \bigoplus X]$. To show that B_1 is a Boolean subalgebra of $[0, \bigoplus X]$, we must show, in view of Theorem 2.10, the following:

(a) $a, b \in B_2 \Rightarrow a \vee^{B_1} b$ exists.
(b) $a, b \in B_1, a \wedge^{B_1} b = 0 \Rightarrow a \perp_{\oplus X} b$.

As for (a), let $a, b \in B_1$. Then $a, b \in B(\bar{X})$, and $a, b \le \bigoplus X$. Since $B(\bar{X})$ is a Boolean subalgebra of $L, a \vee^{B(\bar{X})} b$ exists and therefore $a \vee^{B(\bar{X})} b \le \bigoplus X$ [since $\bigoplus X \in B(\bar{X})$]. This puts a $\vee^{B(\bar{X})} b$ in B_1. Now, if $u \in B_1$ and $a, b \le u$ in B_1, then $a, b \le u$ in $B(\bar{X})$. Hence $a \vee^{B(\bar{X})} b \le u$ and therefore $a \vee^{B_1} b$ exists and equals $a \vee^{B(\bar{X})} b$.

As for (b), let $a, b \in B_1$ and assume that $a \wedge^{B_1} b = 0$. Then $a \wedge^{B(\bar{X})} b = a \wedge^{B_1} b = a \wedge^{B_1} b = 0$. So, as $B(\bar{X})$ is a Boolean subalgebra of

L, Theorem 2.10 implies that $a \perp b$. Hence $a \oplus b = a \vee^{B(\bar{X})} b \leq \bigoplus X$ and therefore $a \perp_{\oplus X} b$.

We have shown that B_1 is a Boolean subalgebra of $[0, \bigoplus X]$, and similarly B_2 is a Boolean subalgebra of $[0, \bigoplus Y]$. This proves the claim. Now apply Theorem 3.14 to get that $B_1 \oplus B_2$ is a Boolean subalgebra of $[0, (\bigoplus X) \oplus (\bigoplus Y)]$. Since $[0, ((\bigoplus X) \oplus (\bigoplus Y))'\}$ is a Boolean subalgebra of $[0, ((\bigoplus X) \oplus (\bigoplus Y))']$, apply Theorem 3.14 again to get that $B_1 \oplus B_2 \oplus \{0, ((\bigoplus X) \oplus (\bigoplus Y))'\}$ is a Boolean subalgebra of $[0, ((\bigoplus X) \oplus (\bigoplus Y)) \oplus ((\bigoplus X) \oplus (\bigoplus Y))'] = [0, 1] = L$. Since $X \cup Y \subseteq B_1 \oplus B_2 \oplus \{0, ((\bigoplus X) \oplus (\bigoplus Y))'\}$, we are done. ■

Combining Lemmas 3.8 and 3.15, we obtain the following result, which we consider the main result of this section.

Theorem 3.16. Let L be an OA and let $X, Y \in J(L)$ be such that $\bigoplus X$ and $\bigoplus Y$ exist, $X \cup \{(\bigoplus X)'\}$, $Y \cup \{(\bigoplus Y)'\} \in J(L)$, $\bigoplus \Delta$ exists for all $\Delta \subseteq X \cup \{(\bigoplus X)'\}$ and $\bigoplus \Gamma$ exists for all $\Gamma \subseteq Y \cup \{(\bigoplus Y)'\}$. Then:

(i) $\bigoplus X \perp \bigoplus Y \Leftrightarrow X \cap Y \subseteq \{0\}$ and $X \cup Y \in J(L)$.
(ii) If, in addition, $\bigoplus (X \cup Y)$ exists, then

$$\bigoplus X \perp \bigoplus Y \Rightarrow \bigoplus (X \cup Y) = (\bigoplus X) \oplus (\bigoplus Y)$$

The following result, which appears as Lemma 2.9 of Foulis and Bennett (1993) and as Lemma 1.4 of Wilce and Feldman (1993), is an immediate consequence of Theorem 3.16.

Corollary 3.17. Let L be an OA and let $F, G \in \mathscr{F}(L) \cap J(L)$. Then

$$\bigoplus F \perp \bigoplus G \Leftrightarrow F \cap G \subseteq \{0\} \quad \text{and} \quad F \cup G \quad \text{is jointly orthogonal}$$

In either case,

$$\bigoplus (F \cup G) = (\bigoplus G) \oplus (\bigoplus G)$$

Proof. The hypothesis that $G, G \in \mathscr{F}(L) \cap J(L)$ implies that $\bigoplus F, \bigoplus G$ exist and $F \cup \{(\bigoplus F)'\}, G \cup \{(\bigoplus G)'\} \in J(L)$, hence $\bigoplus \Delta$ exists $\forall \Delta \subseteq F \cup \{(\bigoplus F)'\}$ and $\bigoplus \Gamma$ exists $\forall \Gamma \subseteq G \cup \{(\bigoplus G)'\}$. Now apply Theorem 3.16. ■

We conclude this section with the following lemma, which will be used in the next section.

Lemma 3.18. Let L be an orthoalgebra and let $X \in J(L)$ be such that $a_\sigma := \bigoplus X$ and $\bigoplus (X \cup \{a'_\sigma\})$ both exist. Then $X \cup \{a'_\sigma\} \in J(L)$ and $\bigoplus (X \cup \{a'_\sigma\}) = 1$.

Proof. First, to show that $X \cup \{a'_\sigma\} \in J(L)$, it is enough to show, using Theorem 3.4, that each $F \in \mathscr{F}(X \cup \{a'_\sigma\})$ is jointly orthogonal. Let then

$F \in \mathscr{F}(X \cup \{a'_\sigma\})$. If $F \subseteq X$, then, by the hypothesis, $f \in J(L)$ and we are done. So we may assume that

$$F = G \cup \{a'_\sigma\} \qquad \text{for some} \quad G \in \mathscr{F}(X)$$

By the hypothesis, $G \in J(L)$, so $\bigoplus G \perp a'_\sigma = \bigoplus \{a'_\sigma\}$ since $\bigoplus G \leq a_\sigma$. Therefore, by Corollary 3.17, $F = G \cup \{a'_\sigma\} \in J(L)$, as desired.

Second, to show that $\bigoplus (X \cup \{a'_\sigma(\}) = 1$, apply part (ii) of Lemma 3.8. ■

4. ORTHOSUMMABLE ORTHOALGEBRAS

To be able to efficaciously define and manipulate orthogonally σ-additive measures or states on orthoalgebras, one needs to define a reasonable notion of σ-orthosummability for orthoalgebras. This was first done by Younce (1987). According to Younce, an orthoalgebra L is σ-orthosummable (or what he calls a σ-orthoalgebra) iff for every countable jointly orthogonal subset $X \subseteq L$ and for all blocks A, B containing X, the supremum $\bigvee^A X$ of X as calculated in A and the supremum $\bigvee^B X$ of X as calculated in B both exist and are equal. This definition seems rather strong and it refers explicitly to the blocks of the orthoalgebra. Thus there has been a need to find a more reasonable notion of σ-orthosummability for orthoalgebras that does not refer explicitly to the blocks of the orthoalgebra and that would be analogous to the well-known notion of σ-orthocompleteness for orthomodular posets as well as to σ-completeness for Boolean algebras (Gudder, 1988; Halmos, 1963; Kalmbach, 1983).

This we do in this section, where we introduce notions of σ-orthosummability and (more generally) orthosummability for orthoalgebras that naturally extend the notions of σ-orthocompleteness and orthocompleteness for orthomodular posets.

About the time this paper was written, Wilce and Feldman (1993) offered another definition of σ-orthosummability (or what they called a σ-orthoalgebra). According to Wilce and Feldman, an orthoalgebra is a σ-orthoalgebra iff every increasing sequence in it has a supremum. It turns out that Wilce and Feldman's notion of a σ-orthoalgebra is equivalent to our notion.

In the sequel, we prove that every chain in an orthosummable orthoalgebra (and every countable chain in a σ-orthoalgebra) has a supremum; and, as a consequence of this, we obtain the result that every chain in an orthocomplete orthomodular poset has a supremum. We also present a proof to a (strong) converse of the latter fact; namely, we show that if P is an orthocomplemented poset in which (i) finite orthogonal suprema exist and (ii) every chain has a supremum, then P is orthocomplete.

Lemma 4.1. Every chain in an OA is jointly compatible.

Proof. Let L be an OA. We first claim that every finite chain in L is jointly compatible. Let $p_0 \leq p_1 \leq \cdots \leq p_n$ be a finite chain in L. We may assume that it is strictly increasing. Consider the difference set

$$D := \{p_0, p_1 - p_0, p_2 - p_1, \ldots, p_n - p_{n-1}\}$$

By Theorem 2.15 of Rüttimann (1989), the set

$$P(D) := \{\bigoplus M: M \subseteq D\} \cup \{(\bigoplus N)': N \subseteq D\}$$

is a sub-OMP of L containing D. Since d is clearly pairwise orthogonal, Corollary 2.11 shows that D is jointly orthogonal. This implies that $\{p_0, \ldots, p_n\}$ is jointly compatible.

Now the lemma follows from the above claim and Theorem 3.6. ■

Definition 4.2. An OA L is called *m-orthosummable* for a cardinal m if for every $X \in J(L)$ with $|X| \leq m$ we have

$$\bigoplus X := \bigvee_{F \in \mathscr{F}(X)} \bigoplus F$$

exists (in L).

The following result is a generalization of the lemma in Holland (1970) to orthoalgebras; its proof, which we omit, is essentially the same as the proof of that lemma.

Lemma 4.3. Let L be an m-orthosummable OA, σ an ordinal number satisfying $|\sigma| < m$, and $\{b_\alpha : \alpha < \sigma\}$ a chain in L that satisfies the following:

(i) $b_0 = 0$.

(ii) β a limit ordinal $< \sigma \Rightarrow \bigvee \{b_\alpha : \alpha < \beta\}$ exists and equals b_β.

Then for every ordinal β satisfying $2 \leq \beta < \sigma$ we have

$$\bigvee \{b_\alpha : \alpha < \beta\} = \bigoplus \{b_{\rho+1} - b_\rho : \rho + 1 < \beta\}$$

The following result is a generalization of the theorem in Holland (1970) to orthoalgebras. The proof which we provide here due to its delicacy is essentially the same as the proof of that theorem.

Theorem 4.4. Every chain of $\leq m$ elements in an m-orthosummable OA has a supremum.

Proof. We proceed by induction. Let L be an m-orthosummable OA and let $\{c_\gamma: \gamma \in \Sigma\}$ be a chain in L with $|\Sigma| \leq m$ and assume that the join of any Σ'-indexed chain exists in L when $|\Sigma'| < |\Sigma|$. Let σ be the least ordinal corresponding to $|\Sigma|$. We may assume that $|\Sigma|$ is infinite and that we have replaced the set Σ by the set $\{\alpha : \alpha < \sigma\}$ so that we are dealing with an ordinal-indexed chain $\{c_\gamma : \gamma < \sigma\}$. If σ is not a limit ordinal, then $\bigvee \{c_\gamma : \gamma < \sigma\} = c_{\sigma - 1}$ and we are done. Thus we may assume that σ is a limit ordinal. In this case, the induction hypothesis implies that

$$b_\alpha := \bigvee \{c_\rho : \rho < \alpha\} \quad \text{exists} \qquad \forall \alpha < \sigma$$

Note that $\alpha \le \beta < \sigma \Rightarrow b_\alpha \le b_\beta$ so that the family $\{b_\alpha : \alpha < \sigma\}$ forms a chain in L; and, moreover, it satisfies the following:

(i) $b_0 = 0$.
(ii) If β is a limit ordinal, then

$$\bigvee \{b_\alpha : \alpha < \beta\} = \bigvee_{\alpha < \beta} \bigvee \{c_\rho : \rho < \alpha\} = \bigvee \{c_\rho : \rho < \beta\} = b_\beta$$

Since $\{b_\alpha : \alpha < \sigma\}$ is a chain, it is jointly compatible by Lemma 4.1. It follows that the pairwise orthogonal set

$$D := \{b_{\alpha+1} - b_\alpha : \alpha + 1 < \sigma\}$$

is jointly orthogonal; therefore $\bigoplus D$ exists, since L is m-orthosummable.

We claim that $\bigvee \{c_\rho : \rho < \sigma\}$ exists and equals $\bigoplus D$. To see this, we first show that $\bigoplus D$ is an upper bound for $\{c_\rho : \rho < \sigma\}$. Indeed, if $\beta < \sigma$, then, since σ is a limit ordinal, we have $\beta + 2 < \sigma$; whence, using Lemma 4.3,

$$\begin{aligned} c_\beta &\le \bigvee \{c_\rho : \rho \le \beta + 1\} = \bigvee \{b_\alpha : \alpha \le \beta + 1\} \\ &= \bigvee \{b_\alpha : \alpha < \beta + 2\} = \bigoplus \{b_{\rho+1} - b_\rho : \rho + 1 \le \beta + 2\} \\ &\le \bigoplus D \end{aligned}$$

Next, we show that $\bigoplus D$ is the least among all such upper bounds. To see this, let $D_\alpha := \{b_{\beta+1} - b_\beta : \beta = 0, 1, \dots, \alpha\}, a < \sigma$. We first establish the following:

$$\bigoplus D_\alpha \le b_{\alpha+1} \qquad \forall \alpha < \sigma \tag{4.1}$$

Indeed, fix $\alpha < \sigma$ and fix a block B that contains the chain $\{b_\gamma : \gamma < \sigma\}$. For every $F \in \mathscr{F}(D_\alpha)$, we have $\bigoplus F = \bigvee^B F \le b_{\alpha+1}$. Hence $\bigoplus D_\alpha = \bigvee_{F \in \mathscr{F}(D_\alpha)} \bigoplus F \le b_{\alpha+1}$, as desired. Now let $u \in L$ and $c_\rho \le u \; \forall \rho < \sigma$. If $F \in \mathscr{F}(D)$, then $\exists \alpha < \sigma$ with $F \subseteq D_\alpha$. Since σ is a limit ordinal, $\alpha + 1 < \sigma$. It follows from (4.1) that

$$\bigoplus F \le \bigoplus D_\alpha \le b_{\alpha+1} = \bigvee \{c_\rho : \rho < \alpha + 1\} \le u$$

which implies that $\bigoplus D = \bigvee_{F \in \mathscr{F}(D)} \bigoplus F \leq u$. Therefore $\bigvee \{c_\rho : \rho < \sigma\}$ exists and equals $\bigoplus D$. This completes the proof of the claim and the induction. ■

Definition 4.5. We call an OA *L-orthosummable* if L is m-orthosummable for every m (or for $m = |L|$). We call a sub-OA A of an orthosummable OA *L-suborthosummable* if for every $X \subseteq A$ with $X \in J(L)$, we have $\bigoplus X \in A$.

An OMP P is called *orthocomplete* (resp., *σ-orthocomplete*) if every (resp., every countable) pairwise orthogonal subset of P has a supremum (in P). Using Corollary 3.5, one can easily show (see Lemma 4.6 below) that an OMP is orthocomplete (resp., σ-orthocomplete) iff it is orthosummable (resp., $\aleph_0$-orthosummable), where $\aleph = |\omega|$. In this sense, our notion of orthosummability (resp., $\aleph_0$-orthosummability) for OAs extends the (well-known) notion of orthocompleteness (resp., σ-orthocompleteness) for OMPs.

It should be noted that Boolean subalgebras of orthosummable orthoalgebras need not be suborthosummable. For example, let

$$\mathbf{B}(\omega) := \{M \subseteq \omega : |M| < \infty \text{ or } |\omega \setminus M| < \infty\}$$

Clearly, $\mathbf{B}(\omega)$ is a Boolean subalgebra of the orthosummable OA $\mathscr{P}(\omega)$. Since the positive even integers form a jointly orthogonal subset of $\mathbf{B}(\omega)$ whose orthogonal sum (which equals its union) does not belong to $\mathbf{B}(\omega)$, $\mathbf{B}(\omega)$ is not suborthosummable.

Lemma 4.6. An OMP is orthocomplete (resp., σ-orthocomplete) iff it is orthosummable (resp., $\aleph_0$-orthosummable).

Proof. ($\Rightarrow$): Assume that P is an orthocomplete OMP. Let $X \in J(P)$. Then $\bigvee X$ exists, $\bigvee F = \bigoplus F$ exists $\forall F \in \mathscr{F}(X)$, and $\bigvee X$ is an upper bound for $\{\bigvee F : F \in \mathscr{F}(X)\}$. Let $u \in P$ be such that $\bigvee F \leq u \ \forall F \in \mathscr{F}(X)$. Then, in particular, $x \leq u \ \forall x \in X$ and so $\bigvee X \leq u$. Thus $\bigoplus X = \bigvee_{F \in \mathscr{F}(X)} (\bigvee F)$ exists and equals $\bigvee X$. Therefore P is orthosummable.

($\Leftarrow$): Assume that P is an orthosummable OMP. Let $X \subseteq P$ be pairwise orthogonal. By Corollary 3.5, $X \in J(P)$; so $\bigvee_{F \in \mathscr{F}(X)} \bigoplus F$ exists. Since P is an OMP, $\bigoplus F = \bigvee F \ \forall F \in \mathscr{F}(X)$. Hence $\bigvee_{F \in \mathscr{F}(X)} \bigoplus F = \bigvee_{F \in \mathscr{F}(X)} \bigvee F = \bigvee X$ exists and P is orthocomplete. ■

Now Theorem 4.4 yields the following result.

Theorem 4.7. Every chain in an orthosummable OA has a supremum.

The following result was mentioned in Navara and Rogalewicz (1991, Proposition 4.5) without proof.

Corollary 4.8. Every chain in an orthocomplete OMP has a supremum.

The following theorem shows that the converse of Corollary 4.8 holds true even for orthoposets that are nonorthomodular.

Theorem 4.9. If P is an orthoposet in which (i) every finite orthogonal set has a supremum, and (ii) every chain has a supremum, then P is orthocomplete.

Proof (Gudder). Let $X \subseteq P$ be an orthogonal set. We may assume that $0 \notin X$. Let S be the set of all suprema that exist for subsets in X. By the hypothesis, any chain in S has an upper bound in S; namely, its supremum. By Zorn's Lemma, S has a maximal element $y_0 = \bigvee Y_0$ for some $Y_0 \subseteq X$.

We claim that $Y_0 = X$. Suppose, contrariwise, that there exists $x \in X \setminus Y_0$. Then $y \leq x'$ $\forall y \in Y_0$, so that $y_0 = \bigvee Y_0 \leq x'$. But then $y_0 \vee x \in S$. Moreover, $y_0 < y_0 \vee x$ since if $y_0 = y_0 \vee x$ $(\geq x)$, then $y_0' \leq x'$; so $1 = y_0 \vee y_0' \leq x'$ and $x = 0$, which is a contradiction. But this, in turn, contradicts the maximality of y_0. Hence $Y_0 = X$ and therefore $y_0 = \bigvee X$ exists, as desired. ■

Corollary 4.10. Every chain in an orthosummable orthoalgebra in which every block is suborthosummable has a supremum in every block containing it, and this supremum is independent of the block containing the chain.

Proof. Note first that it is enough to prove the corollary for an m-orthosummable OA L, where m is any infinite cardinal.

Now a careful examination of the proofs of Lemma 4.3 and Theorem 4.4 and a careful rephrasing of the induction hypothesis reveal that all joins that have been calculated in L can be calculated in any block B containing the given chain. We omit the details since they are essentially the same as the details of the proof of Theorem 4.4. We only point out that the hypothesis that every block of L is suborthosummable will ensure that $\bigoplus D$ (as provided by the proof of Theorem 4.4) belongs to every block containing the given chain. ■

Lemma 4.11. Let L be an orthosummable OA in which every block is suborthosummable and let $X \in J(L)$. Then $\bigvee^B X$ exists in any block B containing X, and $\bigvee^B X = \bigoplus X$.

Proof. Let $X \in J(L)$ and let B be a block containing X. Since L is orthosummable, $\bigoplus X$ exists; and since, by the hypothesis, B is suborthosummable, $\bigoplus X \in B$. Evidently, $\bigoplus X$ is an upper bound for X in B. Let $u \in B$ and $x \leq u$ $\forall x \in X$. Then $\bigoplus F = \bigvee^B F \leq u$ $\forall F \in \mathscr{F}(X)$ and hence $\bigoplus X = \bigvee_{F \in \mathscr{F}(X)} \bigoplus F \leq u$. Therefore $\bigvee^B X$ exists and equals $\bigoplus X$. ■

Theorem 4.12. Let L be an OA and consider the following statements:

1. L is orthosummable and each of its blocks is suborthosummable.
2. L is orthosummable in the sense of Younce.
3. Every block of L is a complete Boolean subalgebra.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2): Let $X \in J(L)$. Since, by (1), L is orthosummable, $\bigoplus X$ exists in L; and, by Lemma 4.11, $\bigvee^B X$ exists and equals $\bigoplus X$ for all blocks B containing X. Thus $\bigvee^B X$ exists in B for every block B containing X and is independent of such a B.

(2) $\Rightarrow$ (3): By (2), every block is an orthocomplete Boolean algebra, and, by the theorem of Holland (1970) (which states that every orthocomplete OML is complete), every such Boolean algebra is complete. ■

Definition 4.13. Let L be an OA. If L is $\aleph_0$-orthosummable, then we say that L is *σ-orthosummable* or, simply, that l is a *σ-orthoalgebra.* We call a sub-OA A of a σ-orthoalgebra an *L sub σ-orthoalgebra* if A is sub $\aleph_0$-orthosummable.

Note that Boolean σ-algebras, unital Boolean σ-rings, σ-complete OMLs, σ-orthocomplete OMPs, and the Wright triangle are all examples of σ-orthoalgebras. Furthermore, the Wright triangle shows that the class of all orthosummable OAs properly contains the class of all orthocomplete OMPs.

Note further that in Theorem 4.12, (3) $\not\Rightarrow$ (1), as Example 3.19 of Habil (1993) shows. Indeed, this example provides an OMP L which is obtained by "pasting" together two disjoint copies L_1 and L_2 of $\mathscr{P}(\mathbf{Z})$ along the "corresponding sections" of all finite or cofinite subsets of L_1 and L_2. As argued in (3.19) of Habil (1993), L is not σ-orthocomplete. Consequently, L is not an $\aleph_0$-orthosummable OA. Therefore L does not satisfy (1) of Theorem 4.12. On the other hand, L_1 and L_2 are the only blocks of L and both are complete. Thus L satisfies (3) of Theorem 4.12.

We do not know at this point whether (2) of Theorem 4.12 implies (1), nor do we know whether every block of an orthosummable (resp., σ-orthosummable) OA is suborthosummable (resp., sub σ-orthosummable).

The following result gives a characterization of σ-orthoalgebras in terms of their countable chains. Moreover, it shows that the converse of the version of Theorem 4.7 involving countable L holds true. However, we do not know at this point whether the full converse of Theorem 4.7 holds true. That is, we do not know whether an OA in which every chain has a supremum is orthosummable.

Theorem 4.14. Let L be an orthoalgebra. The following statements are equivalent:

1. L is a σ-orthoalgebra.
2. Every increasing sequence in L has a supremum (in L); that is, L is a σ-orthoalgebra in the sense of Wilce and Feldman (1993).

Proof. (1) $\Rightarrow$ (2): This part is a consequence of Theorem 4.7.

(2) $\Rightarrow$ (1): Let $\{x_i\}_{i\in\omega}\in J(L)$. Set $s_n := \bigoplus_{i=0}^{n} x_i$ ($n = 0, 1, 2, \ldots$). Evidently, $(s_n)_{n\in\omega}$ is increasing; so, by (2), $\bigvee_{n\in\omega} s_n$ exists.

We claim that $\bigoplus_{i\in\omega} x_i$ exists and equals $\bigvee_{n\in\omega} s_n$. Indeed, notice first that $F\in\mathscr{F}(\{x_i\}_{i\in\omega}) \Rightarrow F\subseteq\{x_0, x_1, \ldots, x_n\}$ for some $n\in\omega \Rightarrow \bigoplus F \le \bigoplus\{x_0, x_1, \ldots, x_n\}$ for some $n\in\omega \Rightarrow \bigoplus F \le \bigvee_{n\in\omega} s_n$. This shows that $\bigvee_{n\in\omega} s_n$ is an upper bound for $\{\bigoplus F\colon F\in\mathscr{F}(\{x_i\}_{i\in\omega})\}$. Second, we show that $\bigvee_{n\in\omega} s_n$ is the least among all such upper bounds. To this end, let $u\in L$ and $\bigoplus F \le u$ $\forall F\in\mathscr{F}(\{x_i\}_{i\in\omega})$. Then, in particular, we have $s_n \le u$ $\forall n\in\omega$. Hence $\bigvee_{n\in\omega} s_n \le u$, and the claim is proved. ■

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